Black Hole Astrophysics
Chapters 9.3

All figures extracted from online sources of from the textbook.

Part I Equation of state

Pressure and Internal energy of various types of gases (Ch 9.3.1~9.3.2)

Introduction

To this stage, we have presented all the conservation laws that would be needed to calculate how plasma behave in a general gravitational field.

$$\left(T^{\alpha\beta}\right)_{\text{gas}} = \begin{pmatrix} \rho c^2 + \varepsilon_g & Q_g^x & Q_g^y & Q_g^z \\ Q_g^x & -2\eta_{v,g}\Sigma^{xx} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{xy} & -2\eta_{v,g}\Sigma^{xz} \\ Q_g^y & -2\eta_{v,g}\Sigma^{yx} & -2\eta_{v,g}\Sigma^{yy} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{yz} \\ Q_g^z & -2\eta_{v,g}\Sigma^{zx} & -2\eta_{v,g}\Sigma^{zy} & -2\eta_{v,g}\Sigma^{zz} - \zeta_{v,g}\Theta + p_g \end{pmatrix}$$

$$Q_g^\alpha = -K_c\left(c^2 P^{\alpha\beta}\nabla_\beta T + T U^\beta \nabla_\beta U^\alpha\right)$$

However, we see above that there are lots of quantities that we don't know yet –

For gases: $\varepsilon_g \; p_g \; K_c \; \eta_{v,g} \; \zeta_{v,g}$ (energy density, pressure, thermal conductivity, viscosity coefficients)

For radiation: $\varepsilon_r \; p_r \; K_r \; \eta_{v,r} \; \zeta_{v,r}$

Therefore, what we do next is to relate them to density ρ and temperature T, and in some cases, plasma composition.

Composition of gases

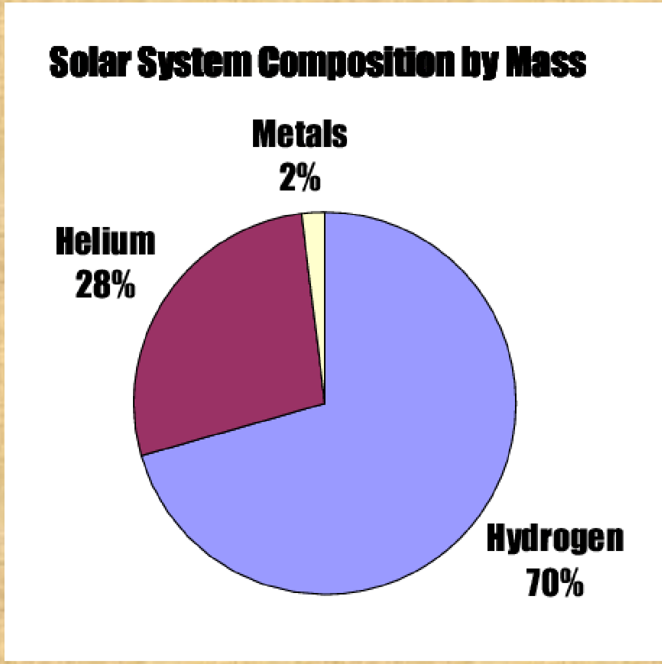
Since the most abundant elements in the universe are Hydrogen and Helium, we usually express the composition of a gas in terms of mass fraction of the elements

X for Hydrogen
Y for Helium
Z for anything heavier (often called "metals")

Unless the gas is exotic (ex electron-positron), the mass fractions sum to 1.

$$X+Y+Z=1$$

	X	Y	Z
Solar Abundance	0.71	0.27	0.02
Early Universe	0.75	0.25	4×10^{-10}



The general distribution function for Thermal gases (9.3.1)

According to statistical mechanics, we can find that gases are distributed in momentum according to (the particle density per unit momentum)

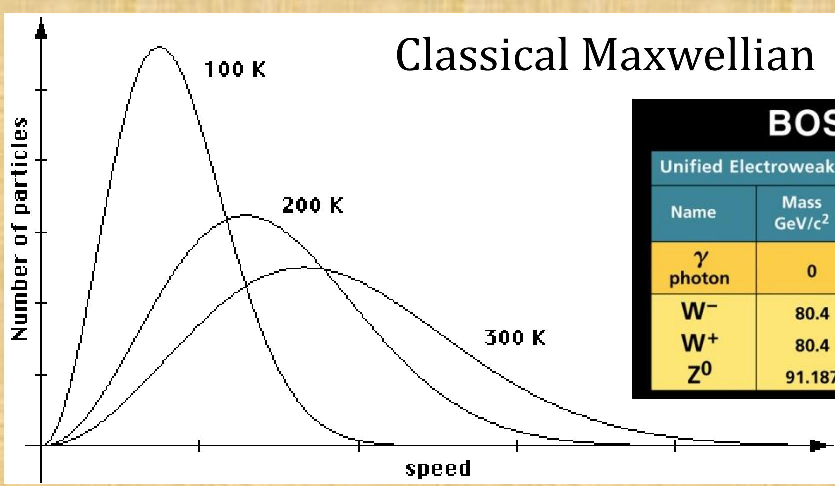
$$\frac{dn}{dp} = \frac{g_s}{h^3} \frac{4\pi p^2}{e^{(\varepsilon(p)-\mu_{chem})/kT} \pm 1}$$

$\varepsilon(p) = \sqrt{p^2c^2 + m_0c^4}$ particle energy; μ_{chem} chemical potential;

$h = 6.62607 \times 10^{-27} \text{erg} \cdot s$ Plank' constant g_s degeneracy factor

$+1$ is for Fermions, half-spin particles ($e^- e^+ p^+ n\, \nu_e \ldots$)

-1 is for Bosons, integer spin particles ($\gamma W^{\pm} Z^0 \ldots$)



BOSONS force carriers spin = 0, 1, 2, ...

Unified Electroweak spin = 1		
Name	Mass GeV/c²	Electric charge
γ photon	0	0
W⁻	80.4	–1
W⁺	80.4	+1
Z⁰	91.187	0

Strong (color) spin = 1		
Name	Mass GeV/c²	Electric charge
g gluon	0	0

Color Charge
Each quark carries one of three types of "strong charge," also called "color charge." These charges have nothing to do with the colors of visible light. There are eight possible

FERMIONS matter constituents spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2		
Flavor	Mass GeV/c²	Electric charge
ν_e electron neutrino	<1×10⁻⁸	0
e electron	0.000511	–1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	–1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	–1

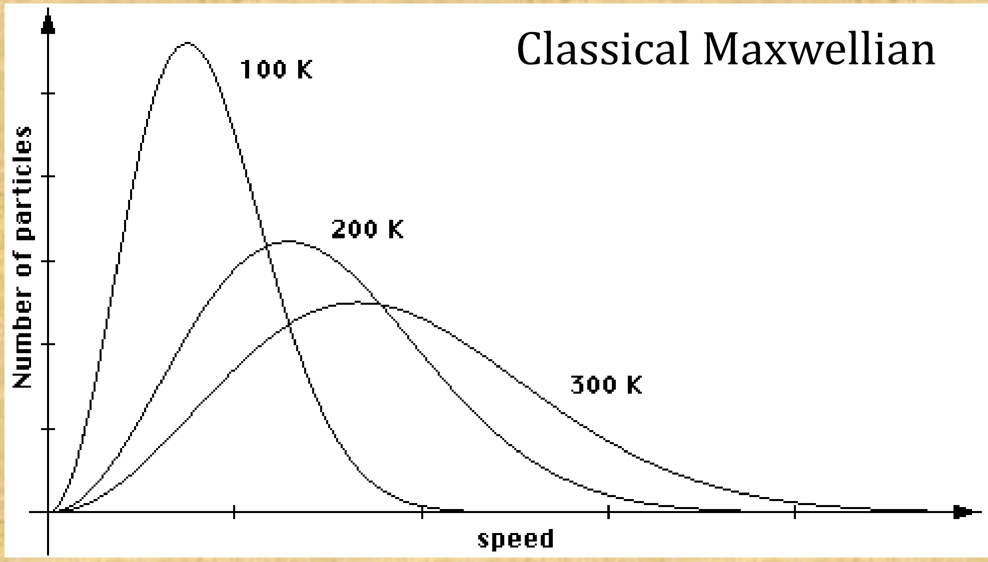
Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c²	Electric charge
u up	0.003	2/3
d down	0.006	–1/3
c charm	1.3	2/3
s strange	0.1	–1/3
t top	175	2/3
b bottom	4.3	–1/3

Determining Energy and Momentum from the distribution function

Since the distribution function $\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{g_s}{h^3} \frac{4\pi \mathrm{p}^2}{e^{(\varepsilon(p)-\mu_{\mathrm{chem}})/\mathrm{kT}} \pm 1}$ tells us how many particles (per unit volume) are contained within a momentum interval,

The total kinetic energy is simply to sum over that of each momentum interval

$$\varepsilon_i = \int \varepsilon_K(p) \frac{\mathrm{dn}_i}{\mathrm{dp}} \mathrm{dp}$$



And the pressure, being momentum flux as we discussed last week, is

$$\mathcal{p}_i = \frac{1}{3} \int p \left(v \frac{\mathrm{dn}_i}{\mathrm{dp}} \right) \mathrm{dp}$$

Particle flux

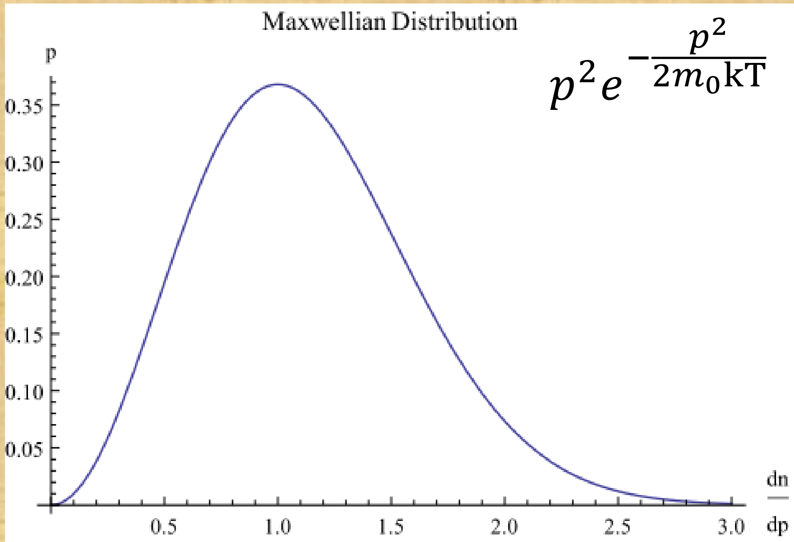
Non-Relativistic Ideal Gas: Tenuous, Warm Fermions (Ch9.3.1.1)

For fermions at not too high a density, the chemical potential is very negative.
And for non-relativistic gases, $\varepsilon \approx m_0c^2 + \varepsilon_K >> \mathrm{kT}$

Thus, the distribution function reduces to

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{g_s}{h^3}\frac{4\pi \mathrm{p}^3}{e^{(\varepsilon(p)-\mu_{\mathrm{chem}})/\mathrm{kT}}+1} \approx \frac{8\pi}{h^3}e^{(\mu_{\mathrm{chem}}-m_0c^2)/\mathrm{kT}}p^2e^{-\frac{\varepsilon_K}{\mathrm{kT}}}$$

$$\approx \frac{8\pi}{h^3}e^{(\mu_{\mathrm{chem}}-m_0c^2)/\mathrm{kT}}p^2e^{-\frac{p^2}{2m_0\mathrm{kT}}}$$



Evaluating the internal energy and pressure, we find the very familiar formulas:

$$\varepsilon_i = \frac{3}{2}\mathrm{nkT}$$

$$\wp_g = \mathrm{nkT}$$

The adiabatic index $\Gamma = \frac{5}{3}$

The polytropic index $n = \frac{3}{2}$

Specific heats $C_p = \frac{5}{2}R \quad C_v = \frac{3}{2}R$

Pressure for different non-relativistic ideal gas compositions

$$\mathcal{P}_g = \text{nkT} = \frac{\rho\text{kT}}{\mu}$$

μ is the mean molecular weight, expressed in units of grams per mole

Gas Type	μ	μ(Solar)
Neutral Hydrogen Gas	1	
Fully Ionized hydrogen gas	0.5	
General Composition Neutral Gas	$\frac{1}{X + 0.25Y + 0.06Z}$	
General Composition Fully Ionized Gas	$\frac{1}{2X + 0.75Y + 0.56Z}$	0.61 $\mathcal{P}_g = 1.63\rho\text{kT}$

Simple explanation for mean molecular weight:

$$\mu = \frac{M_{\text{total}}}{N_H + N_{\text{He}} + N_{\text{Metal}}} = \frac{M_{\text{tot}}}{M_{\text{tot}} \cdot X + M_{\text{tot}} \cdot Y/4 + M_{\text{metal}} \cdot Z/m_{\text{metal}}} = \frac{1}{X + 0.25Y + Z/m_{\text{metal}}}$$

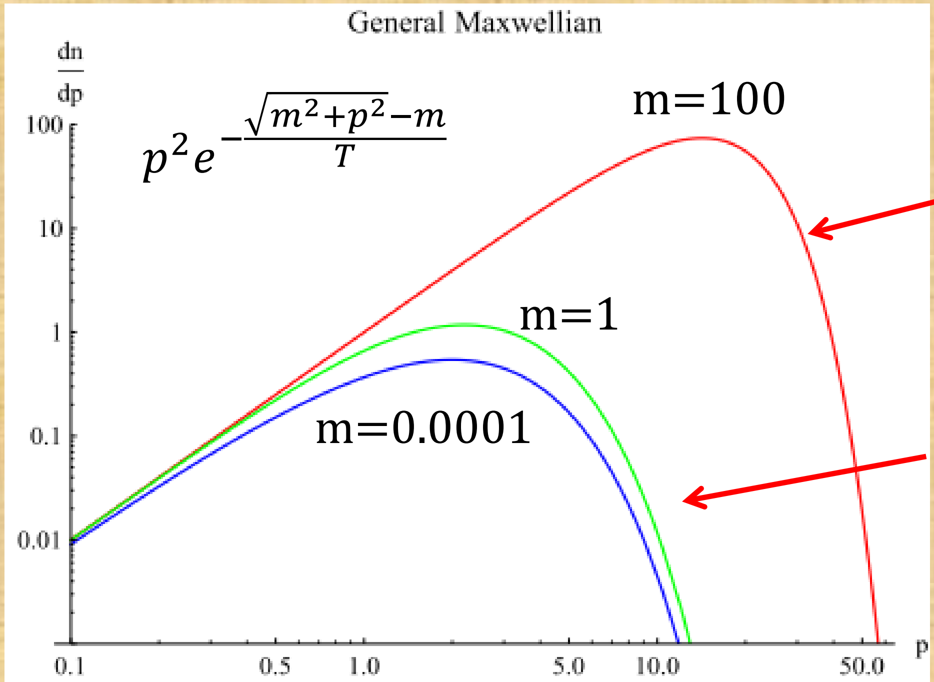
$$M_{\text{tot}} \cdot X = M_H = N_H \cdot 1\,;\; M_{\text{tot}} \cdot Y = M_{\text{He}} = N_{\text{He}} \cdot 4\,; M_{\text{tot}} \cdot Z = M_{\text{metal}} = N_{\text{metal}} \cdot m_{\text{metal}}$$

Relativistic Ideal Gas: Tenuous, Hot Fermions (Ch9.3.1.2)

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{g_s}{h^3}\frac{4\pi p^3}{e^{(\varepsilon(p)-\mu_{\text{chem}})/kT} \pm 1} \approx \frac{8\pi}{h^3} e^{(\mu_{\text{chem}}-m_0c^2)/kT} p^2 e^{-\frac{\varepsilon_K}{kT}}$$

Since for general situations, the kinetic energy is $\varepsilon_K = \sqrt{(m_0c^2)^2 + (pc)^2} - m_0c^2$
This changes the distribution to

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{8\pi}{h^3} e^{(\mu_{\text{chem}}-m_0c^2)/kT} p^2 e^{-\frac{\sqrt{(m_0c^2)^2+(pc)^2}-m_0c^2}{kT}}$$



Classical Maxwellian

Shape of cut-off is affected by the mass

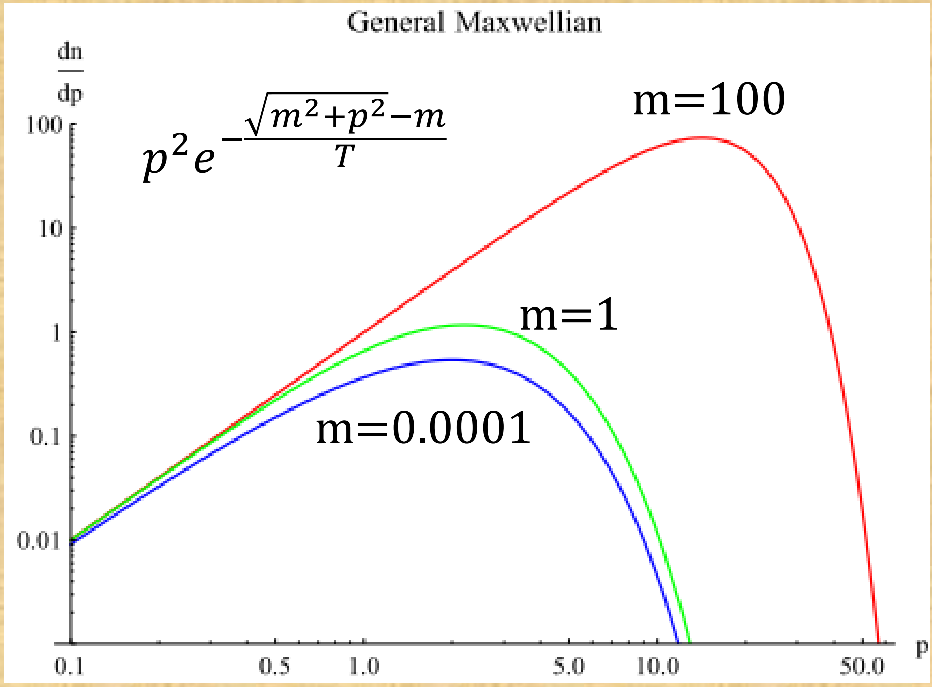
The highly relativistic case

When the kinetic energy is much greater than the rest mass energy, it is mainly dominated by the pc term.

$$\frac{\text{dn}}{\text{dp}} = \frac{8\pi}{h^3} e^{(\mu_{\text{chem}} - m_0 c^2)/\text{kT}} p^2 e^{-\frac{pc}{\text{kT}}}$$

Evaluating the internal energy and pressure, we find the very familiar formulas:

$$\varepsilon_i = 3\text{nkT} \;\; \wp_g = \text{nkT}$$



The adiabatic index $\Gamma = \frac{4}{3}$

The polytropic index $n = 3$

Specific heats $C_p = 4R \;\; C_v = 3R$

As we would expect, this will turn out to be very much the same as photons since photons have rest mass and their energies are only kinetic.

Photon Gas: Hot Bosons (Ch 9.3.1.3)

Taking the distribution for photons and using the fact that $\varepsilon_K = \varepsilon = pc = h\nu$ and $g_s = 2$ for two polarization states

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{g_s}{h^3}\frac{4\pi p^2}{e^{(\varepsilon(p)-\mu_{\mathrm{chem}})/kT}-1} = \frac{8\pi}{h^3}\frac{p^2}{e^{pc/kT}-1}$$

If we look at the spectral energy distribution, we see that it should be very familiar

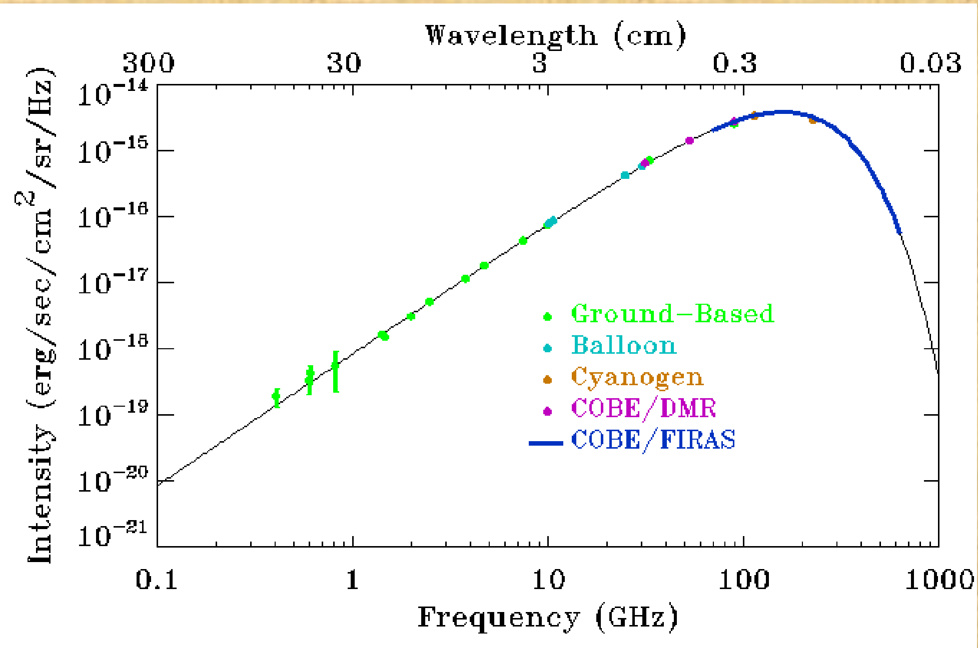
$$\varepsilon\frac{\mathrm{dn}}{\mathrm{d}\nu} = \frac{8\pi h\nu}{c^3}\frac{\nu^2}{e^{h\nu/kT}-1}$$

It is simply the Plankian SED !

As for the intensity,

$$I(\nu) = \frac{c}{4\pi}\varepsilon\frac{\mathrm{dn}}{\mathrm{d}\nu} = \frac{2h\nu}{c^2}\frac{\nu^2}{e^{h\nu/kT}-1} = B_\nu(T)$$

This is also the out familiar form of the Plank function that describes the intensity per unit frequency. (Black Body Distribution)



Energy and Pressure for a Photon gas

Evaluating the internal energy and pressure, we find:

$$\varepsilon_r = 3\wp_g = \mathrm{a}\mathrm{T}^4$$

$$\mathrm{a} = 7.56577 \times 10^{-15}\mathrm{erg} \cdot \mathrm{cm}^{-3}K^{-4}$$

This gives:
The adiabatic index $\Gamma = \frac{4}{3}$
The polytropic index $n = 3$
Specific heats $C_p = 4R \;\; C_v = 3R$

Which is the same as a relativistic Fermion gas.

Denerate Gas: Dense Fermions (Ch9.3.1.4)

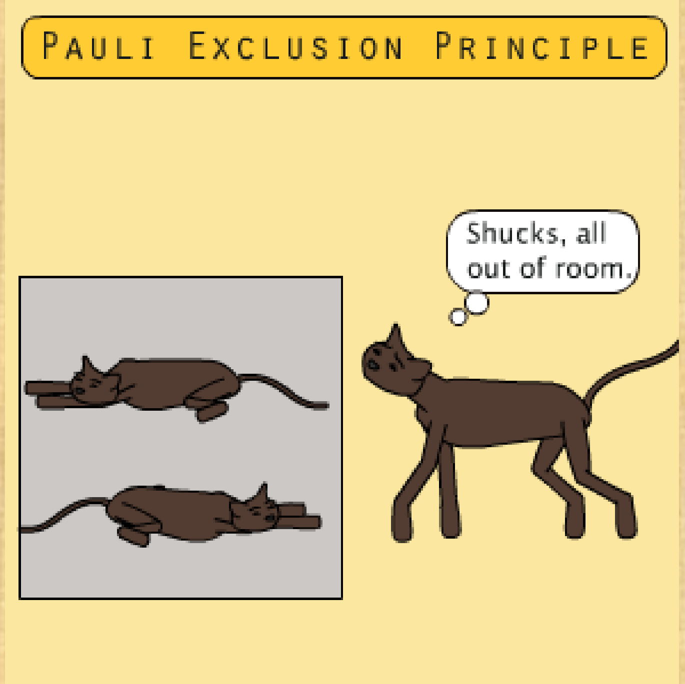
Previously, we have discussed cases where the chemical potential is very negative and therefore causes the exponential term to be much larger than 1.

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{g_s}{h^3}\frac{4\pi \mathrm{p}^2}{e^{(\varepsilon(p)-\mu_{\mathrm{chem}})/\mathrm{kT}} \underline{\pm 1}}$$

However, when the density of Fermions, for example, becomes so high that the Pauli Exclusion Principle can't be neglected, then the '1' in the denominator becomes important.

$$\frac{\mathrm{dn}}{\mathrm{dp}} = \frac{8\pi}{h^3}\frac{p^2}{e^{(\varepsilon(p)-\mu_{\mathrm{chem}})/\mathrm{kT}} + 1}$$

$$\frac{\mathrm{dn}}{\mathrm{d}\varepsilon_K} = \frac{8\pi}{h^3 c^3}\sqrt{{\varepsilon_K}^2 + 2\varepsilon_K m_0 c^2}\,\frac{\varepsilon_K + m_0 c^2}{e^{\frac{\varepsilon_K + m_0 c^2 - \mu_{\mathrm{chem}}}{\mathrm{kT}}} + 1}$$



How to define “degenerate”?

In our introduction to degenerate gases, we noted that for dense fermions, the +1 must be considered.
It should then be obvious that the exponential term can’t be too large.

$$\frac{\mathrm{dn}}{\mathrm{d}\varepsilon_K} = \frac{8\pi}{h^3c^3}\sqrt{{\varepsilon_K}^2 + 2\varepsilon_K m_0 c^2}\,\frac{\varepsilon_K + m_0c^2}{e^{\frac{\varepsilon_K + m_0c^2 - \mu_{\mathrm{chem}}}{\mathrm{kT}}} + 1}$$

To be more precise, we can define a “Fermi Temperature” $T_F = \frac{\varepsilon_F}{\mathrm{k}} = \frac{\mu_{\mathrm{chem}} - m_0c^2}{k}$

The exponential then becomes $e^{\frac{\varepsilon_K - \varepsilon_F}{\mathrm{kT}}}$.

Now, we see that it is clear that there are two cases:

1. $\varepsilon_K \gg \varepsilon_F$: The exponential term is large, we have a non-degenerate gas.

2. $\varepsilon_K \ll \varepsilon_F$: The exponential term is small. A degenerate gas.

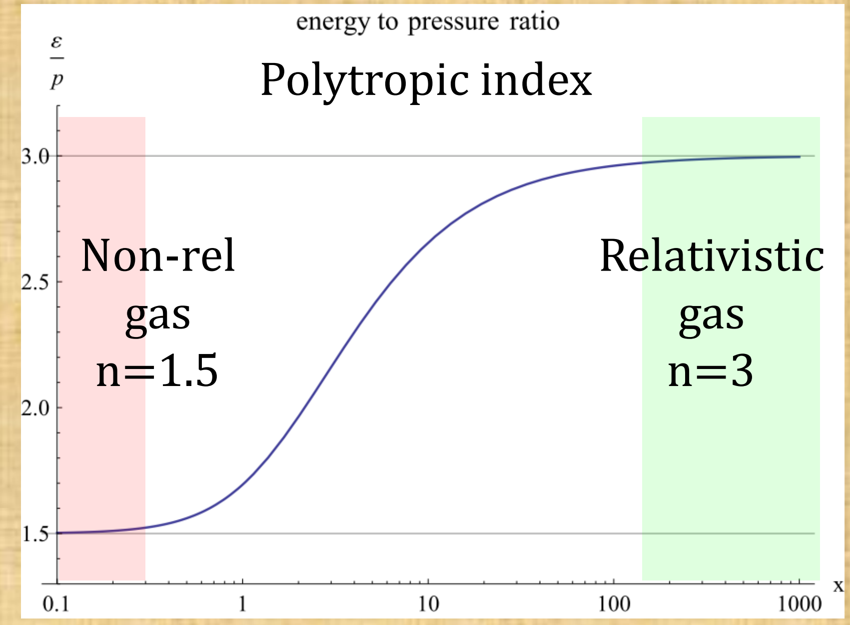
Pressure and Energy

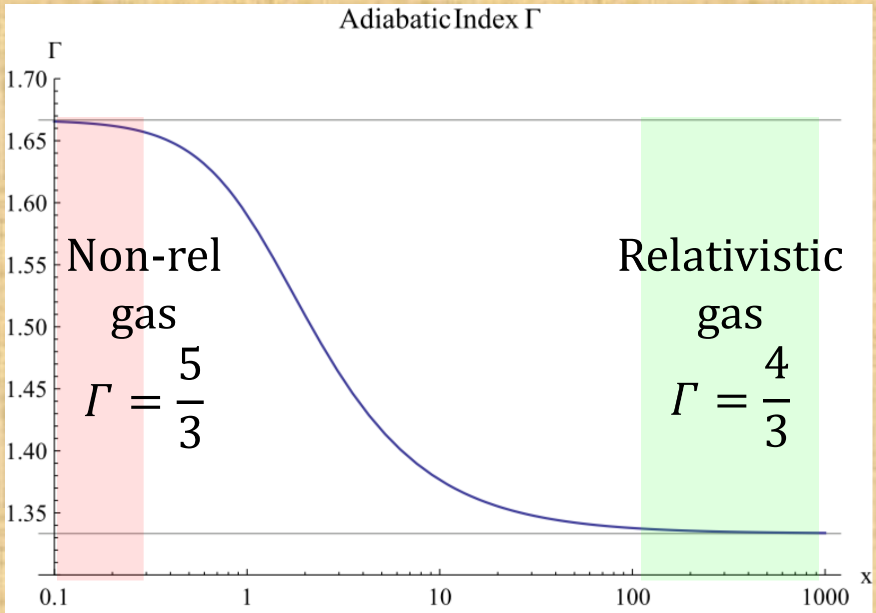
Evaluating the pressure and energy, we get:

$$p = \frac{8\pi}{3}\left(\frac{m_0 c}{h}\right)^3 m_0 c^2 P(x) \qquad \varepsilon = \frac{8\pi}{3}\left(\frac{m_0 c}{h}\right)^3 m_0 c^2 E(x) \qquad x \equiv \frac{\varepsilon_F}{m_0 c^2}$$

With the normalized energy and pressure functions:

$$P(x) = x(2x^2 - 3)\sqrt{x^2 + 1} + 3\sinh^{-1} x$$
$$E(x) = 3x(2x^2 + 1)\sqrt{x^2 + 1} - 8x^3 - 3\sinh^{-1} x$$





Some handy numbers

Handy expressions for the pressure for a degenerate electron gas are, for the non-relativistic and relativistic cases,

$$p_{\mathrm{e,NR}} = 1.00 \times 10^{13}\,\mathrm{dyn\,cm^{-2}} \left(\frac{\rho}{\mu}\right)^{5/3}$$

$$p_{\mathrm{e,R}} = 1.24 \times 10^{15}\,\mathrm{dyn\,cm^{-2}} \left(\frac{\rho}{\mu}\right)^{4/3}$$

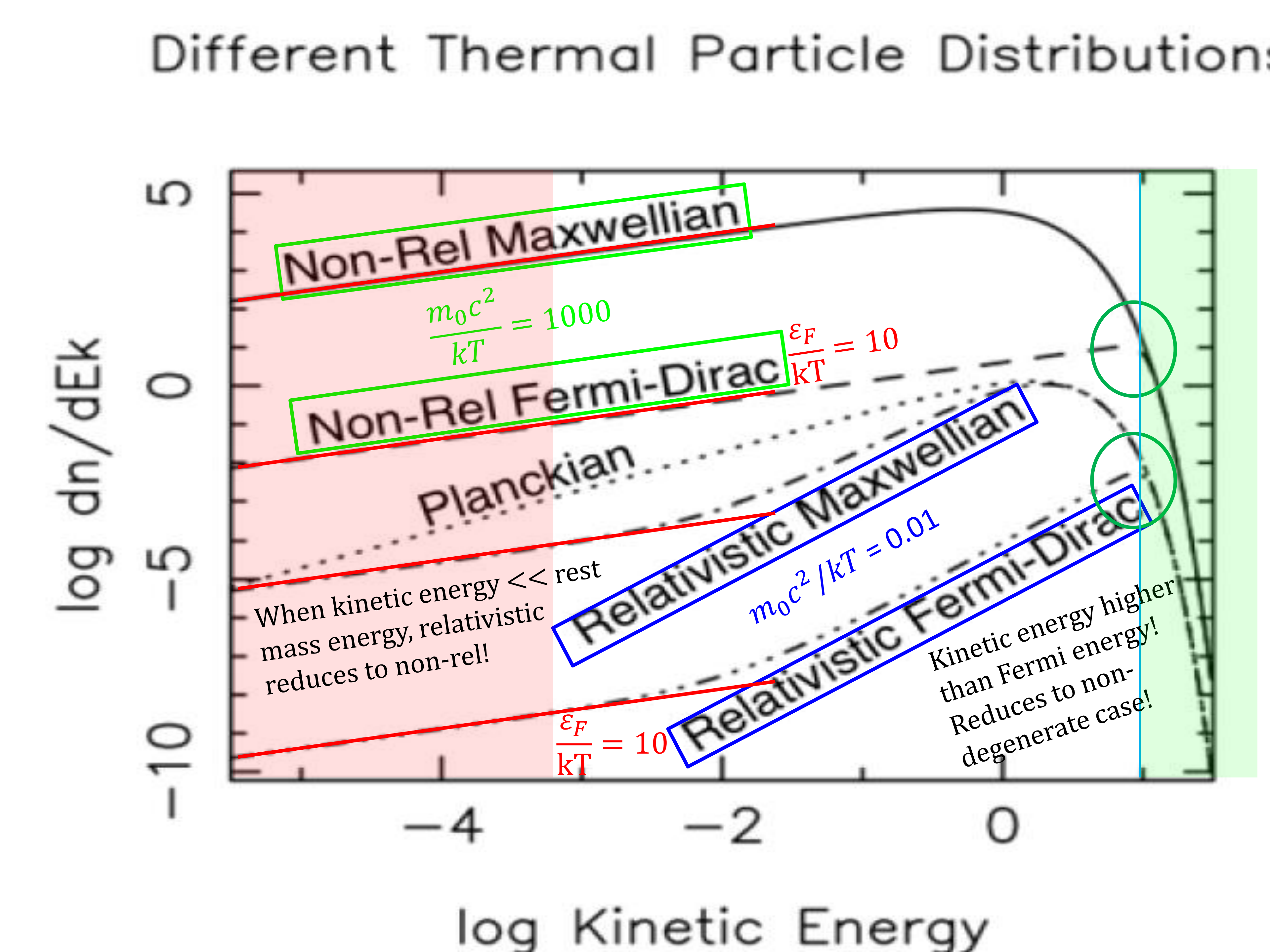
and for a degenerate neutron gas

$$p_{\mathrm{n,NR}} = 5.50 \times 10^{9}\,\mathrm{dyn\,cm^{-2}}\,\rho^{5/3}$$

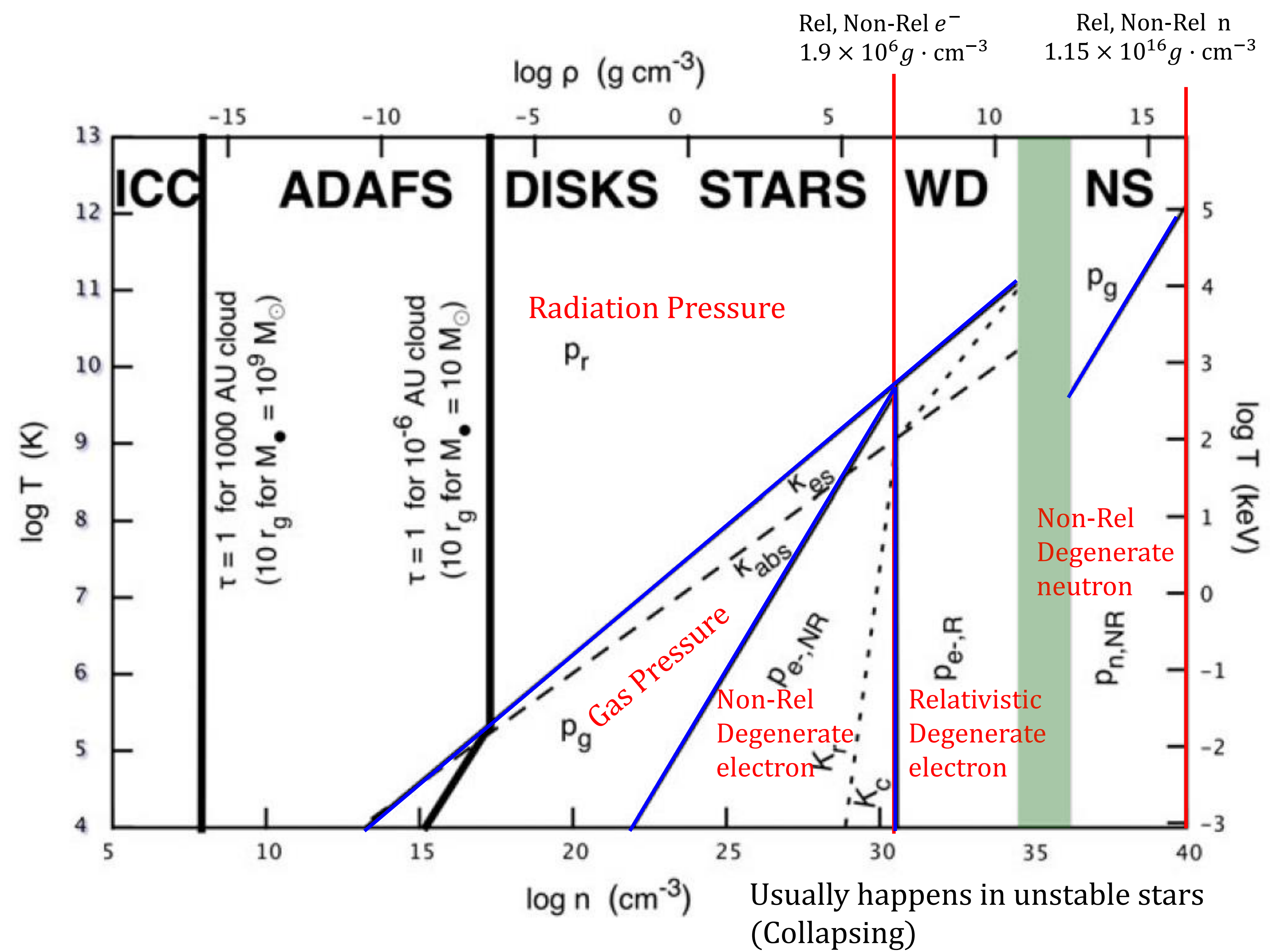
$$p_{\mathrm{n,R}} = 1.24 \times 10^{15}\,\mathrm{dyn\,cm^{-2}}\,\rho^{4/3}$$

with ρ and μ in cgs units, and the standard $\varepsilon = p/(\Gamma - 1)$ giving the internal energy density for each. Note the similarity between the two different degenerate gases in the relativistic cases.

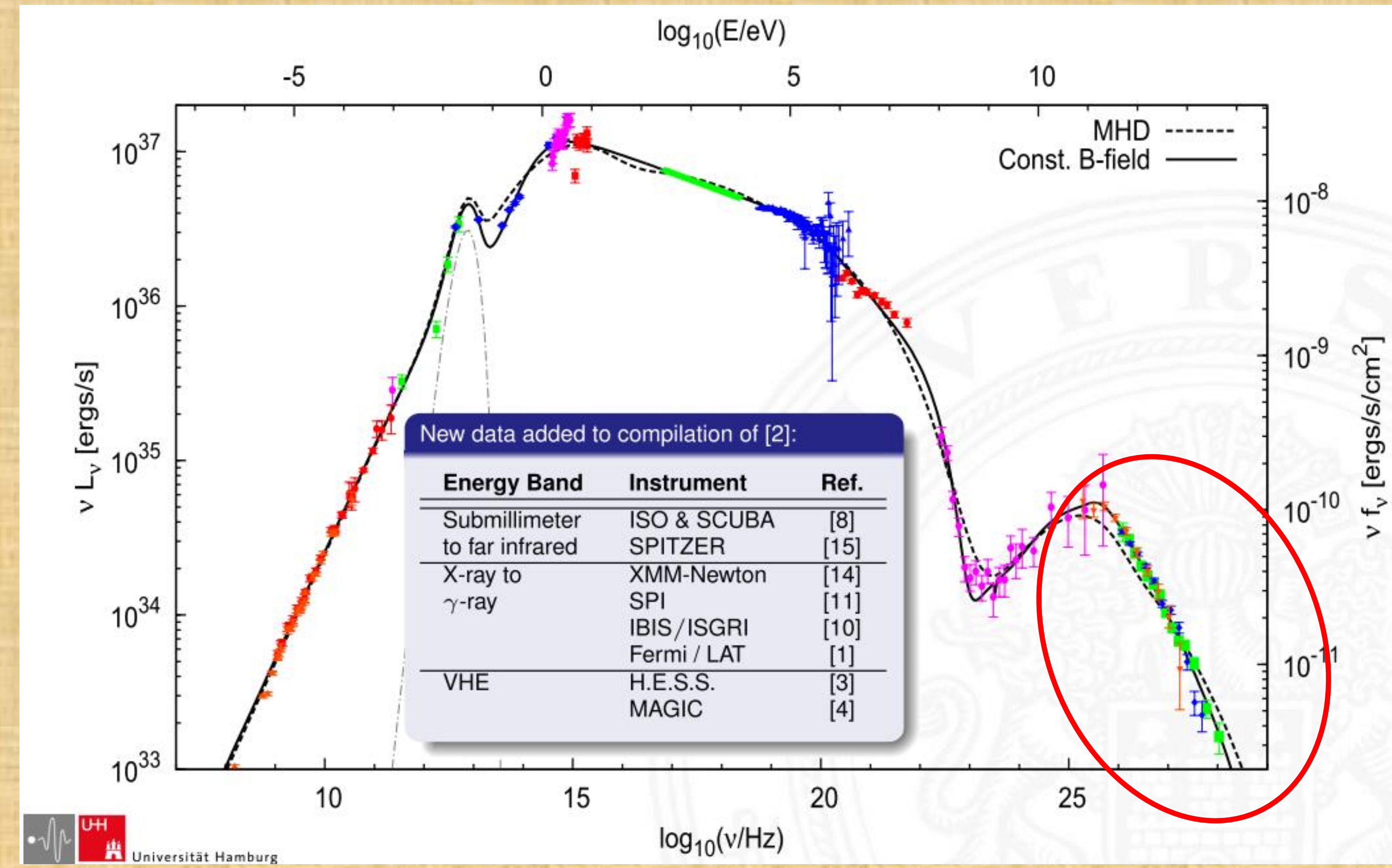
The boundaries between the non-relativistic and relativistic cases are approximately
$1.9 \times 10^{6} g \cdot \mathrm{cm}^{-3}$ for the degenerate electron gas and
$1.15 \times 10^{16} g \cdot \mathrm{cm}^{-3}$ for degenerate neutrons.



Different Thermal Particle Distributions
Non-Rel Maxwellian
$\frac{m_0c^2}{kT} = 1000$
Non-Rel Fermi-Dirac
$\frac{\varepsilon_F}{kT} = 10$
Planckian
Relativistic Maxwellian
$m_0c^2/kT = 0.01$
Relativistic Fermi-Dirac
$\frac{\varepsilon_F}{kT} = 10$
When kinetic energy << rest mass energy, relativistic reduces to non-rel!
Kinetic energy higher than Fermi energy! Reduces to non-degenerate case!
log dn/dEk
5
0
−5
−10
−4
−2
0
log Kinetic Energy


Rel, Non-Rel e^-
$1.9 \times 10^6 g \cdot \mathrm{cm}^{-3}$
Rel, Non-Rel n
$1.15 \times 10^{16} g \cdot \mathrm{cm}^{-3}$
log ρ (g cm^{-3})
−15
−10
−5
0
5
10
15
ICC
ADAFS
DISKS
STARS
WD
NS
τ = 1 for 1000 AU cloud
(10 r_g for $M_\bullet = 10^9 M_\odot$)
τ = 1 for 10^{-6} AU cloud
(10 r_g for $M_\bullet = 10 M_\odot$)
Radiation Pressure
p_r
κ_{es}
κ_{abs}
$p_{e-,NR}$
$p_{e-,R}$
$p_{n,NR}$
p_g
Gas Pressure
p_g
Non-Rel
Degenerate
electron
κ_r
κ_c
Relativistic
Degenerate
electron
Non-Rel
Degenerate
neutron
log T (K)
13
12
11
10
9
8
7
6
5
4
log T (keV)
5
4
3
2
1
0
-1
-2
-3
5
10
15
20
25
30
35
40
log n (cm^{-3})
Usually happens in unstable stars
(Collapsing)

Nonthermal gases (Ch 9.3.2)



Possibly due to Fermi acceleration in the universe, many sources exhibit a powerlaw spectrum in the high energy end. The Crab Nebula is given as an example to the left. (Radio lobes, jets often also show this behavior)

This is usually called non-thermal since particles that emit this radiation must have energies way higher than the thermal value 'kT'. Lorentz factors can go even up to 10^6 or higher.

Power law spectra

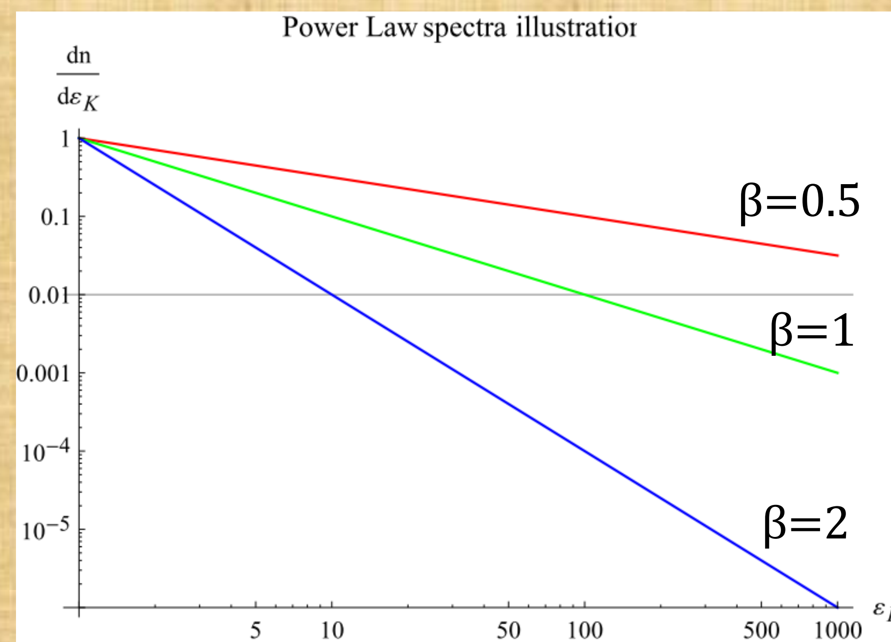
For such cases, it is common to assume that the particles distribute in energy as a power law shape:

$$\frac{\mathrm{dn}}{\mathrm{d}\varepsilon_K} = n\underbrace{\left(\frac{1-\beta}{\varepsilon_{K,\mathrm{Max}}^{1-\beta} - \varepsilon_{K,\mathrm{min}}^{1-\beta}}\right)}_{\text{normalization}}\varepsilon_K^{-\beta}$$

Which energies $\varepsilon_{K,\mathrm{min}} < \varepsilon_K < \varepsilon_{K,\mathrm{Max}}$

If $\beta > 1$, then the distribution function is steep and dominated by low-energy particles, perhaps even a very low-energy thermal distribution. On the other hand,

If $\beta < 1$, then the distribution is shallow, dominated by the high-energy end, and must be cut off more steeply beyond $\varepsilon_{K,\mathrm{Max}}$.



Energy and Pressure for non-thermal particles

Evaluating the energy and pressure for non-thermal particles, we find that

$$\varepsilon = 3p = n\left(\frac{1-\beta}{2-\beta}\right)\left(\frac{\varepsilon_{K,\mathrm{Max}}^{2-\beta} - \varepsilon_{K,\mathrm{min}}^{2-\beta}}{\varepsilon_{K,\mathrm{Max}}^{1-\beta} - \varepsilon_{K,\mathrm{min}}^{1-\beta}}\right)$$

This gives a Adiabatic Index $\Gamma = \frac{4}{3}$, same as for highly relativistic particles. (This should be trivial since by origin, they are highly relativistic)

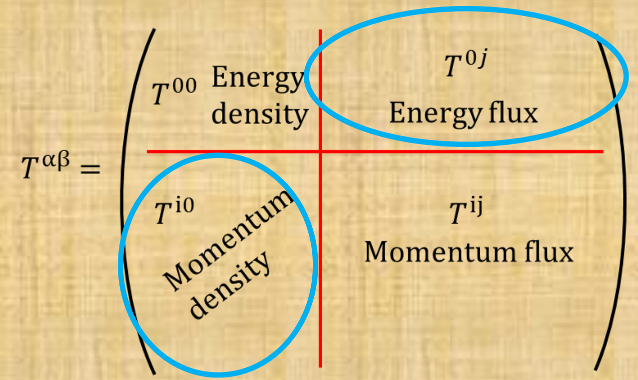
Part II Equation of state

Conductivity and Viscosity (Ch9.3.3~9.3.7)

Thermal Conductivity (Ch9.3.3)

Recall From last week

With the knowledge that $\vec{Q}_g = -K_c \vec{\nabla} T$ and that it corresponds to the T^{i0} and T^{0j} terms, we could guess that in locally flat space-time, the components would read as



$$\left(T^{\alpha\beta}\right)_{\text{Conduction}} = \begin{pmatrix} 0 & Q_g^x & Q_g^y & Q_g^z \\ Q_g^x & 0 & 0 & 0 \\ Q_g^y & 0 & 0 & 0 \\ Q_g^z & 0 & 0 & 0 \end{pmatrix}$$

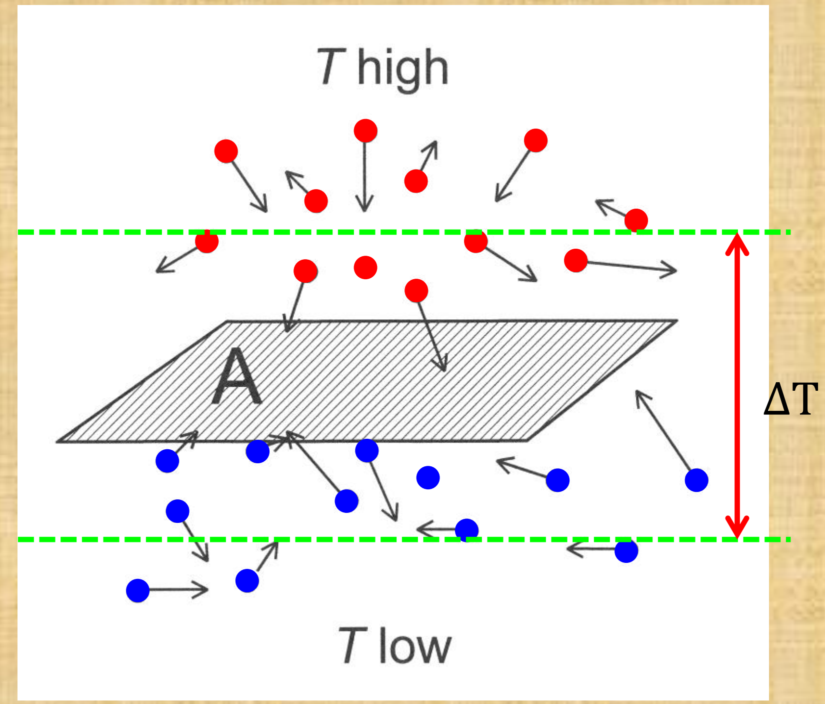
However, we can see that Q_g is actually still a 3-vector and the above form is simply from an educated guess. Therefore we need to first rewrite Q_g into a 4-vector Q_g^α.

We find that it can be expressed as

$$Q_g^\alpha = -K_c\left(c^2 P^{\alpha\beta}\nabla_\beta T + T U^\beta \nabla_\beta U^\alpha\right) \text{ with } P^{\alpha\beta} = \frac{1}{c^2}U^\alpha U^\beta + g^{\alpha\beta}$$

Or, $\vec{Q}_g = -K_c\left(c^2 \overleftrightarrow{P} \cdot \vec{\nabla} T + T\,\vec{U} \cdot \vec{\nabla}\,\vec{U}\right)$ with $\overleftrightarrow{P} = \frac{1}{c^2}\,\vec{U} \otimes \vec{U} + \left(\overleftrightarrow{g}\right)^{-1}$

A simple kinetic picture



Consider a picture like the one on the left.

If we consider that a pair of particles are exchanged, then there will be a net energy transfer from top to bottom.

Therefore we can write heat flux as (particle number flux)×(energy difference)

For a thermal gas, the energy that is required to heat it by ΔT is $\Delta E = C_V \Delta \mathrm{T}$.

In terms of differential quantities, we can write $\Delta \mathrm{T} = \ell_c \frac{\mathrm{dT}}{\mathrm{dz}}$

Putting it all together, we get $\mathrm{Q} \approx -\frac{1}{3} n \langle V_c \rangle C_V \ell_c \frac{\mathrm{dT}}{\mathrm{dz}}$.

Comparing with $\vec{Q} = -K_c \vec{\nabla} T$, we find the diffusion coefficient $K_c \approx -\frac{1}{3}(C_V n)\langle V_c \rangle \ell_c$

Q1.Why is ℓ_c the mean free path?

Q2.Why is it C_V?

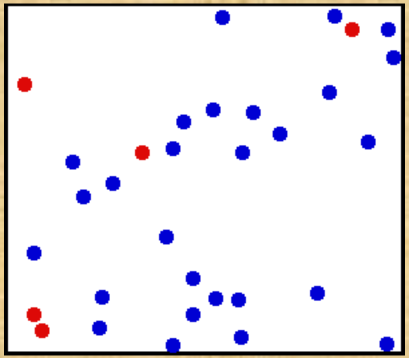
http://en.wikipedia.org/wiki/Thermal_conductivity

Thermal Conductivity

As we have just found, the thermal condutivity is equal to $K_c \approx \frac{1}{3}(nC_V)\langle V_c \rangle \ell_c$

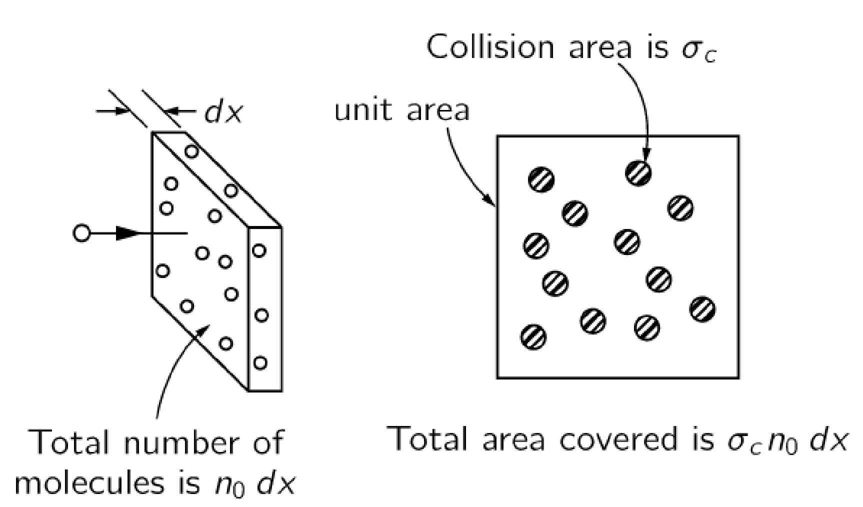
For thermal conduction in a electron-ion plasma, it would be sufficient to only consider electrons since they are fast.

For a classical thermal gas, $nC_V = \frac{3}{2} n_e k$



The rms velocity is $\langle V_c \rangle = \sqrt{\frac{3\mathrm{kT}}{m_e}}$

The mean free path, by definition is the inverse of the density multiplied by the collision cross-section. $\ell_c = \frac{1}{n_e \sigma_c}$



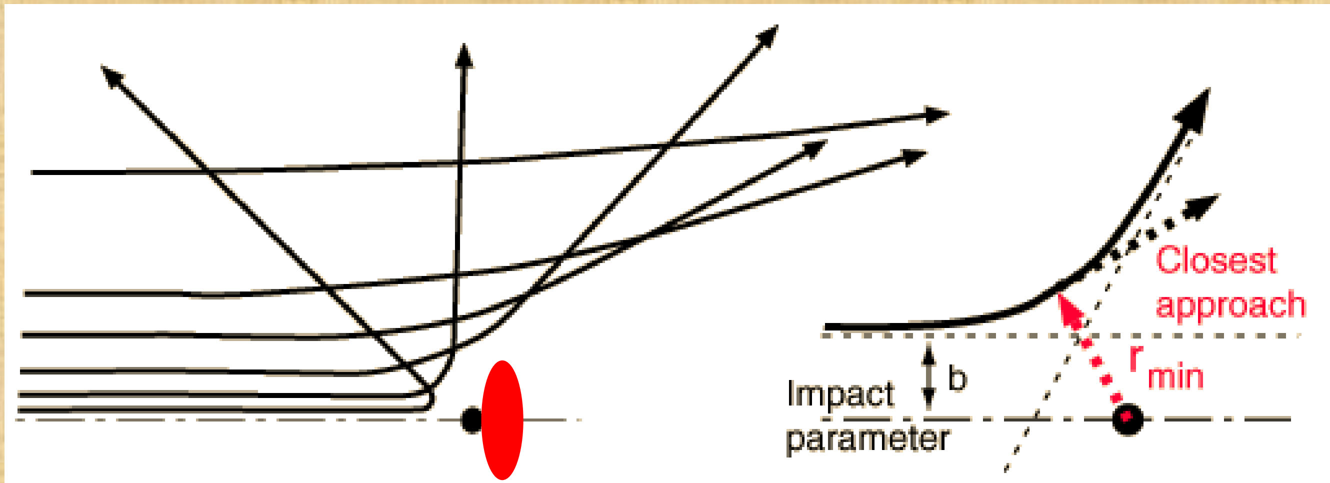
Determining the mean free path

The mean free path $\ell_c = \frac{1}{n_e \sigma_c}$

The easiest was to estimate the collision cross-section is to give it a radius, thus, $\sigma_c = \pi r_c{}^2$

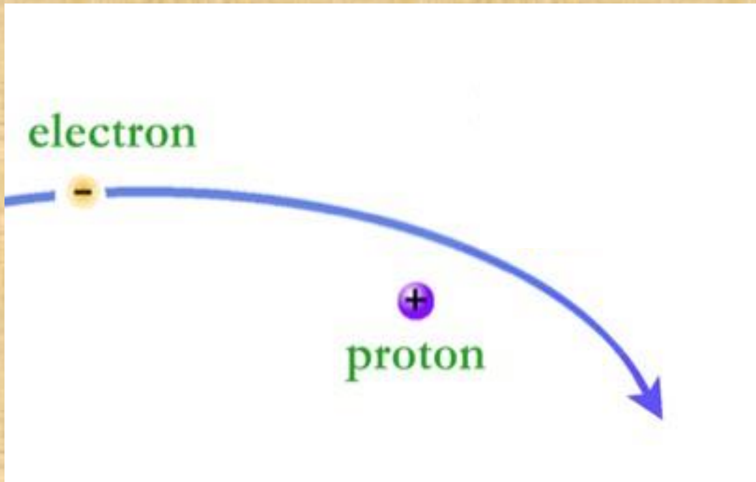
Therefore the actual problem is to find some reasonable radius to apply into the formula. (This was actually already discussed in Ch1 of plasma Astrophys.)

My own idea is like this: Since the mean free path is the distance of which a particle travels before crashing into something and thereby changing direction of motion, the cross-section associated with it would be defined by some radius within which the injected particle would be deflected by a large angle. (red oval below)



http://en.wikipedia.org/wiki/Coulomb_collis

Determining the mean free path



For coulomb collisions, if the particle looses most of its initial kinetic energy to the coulomb field, then it now no longer knows which direction it came from.

The radial coulomb field then changes its direction according to how close the particle is.

Thus, we can approximate the radius by equating the thermal kinetic energy and the Coulomb potential energy.

$$\varepsilon_C = \frac{e^2}{r_c} = \mathrm{kT} = \varepsilon_K$$

This then give us a classical Coulomb collision radius

$$r_c = \frac{e^2}{\mathrm{kT}}$$

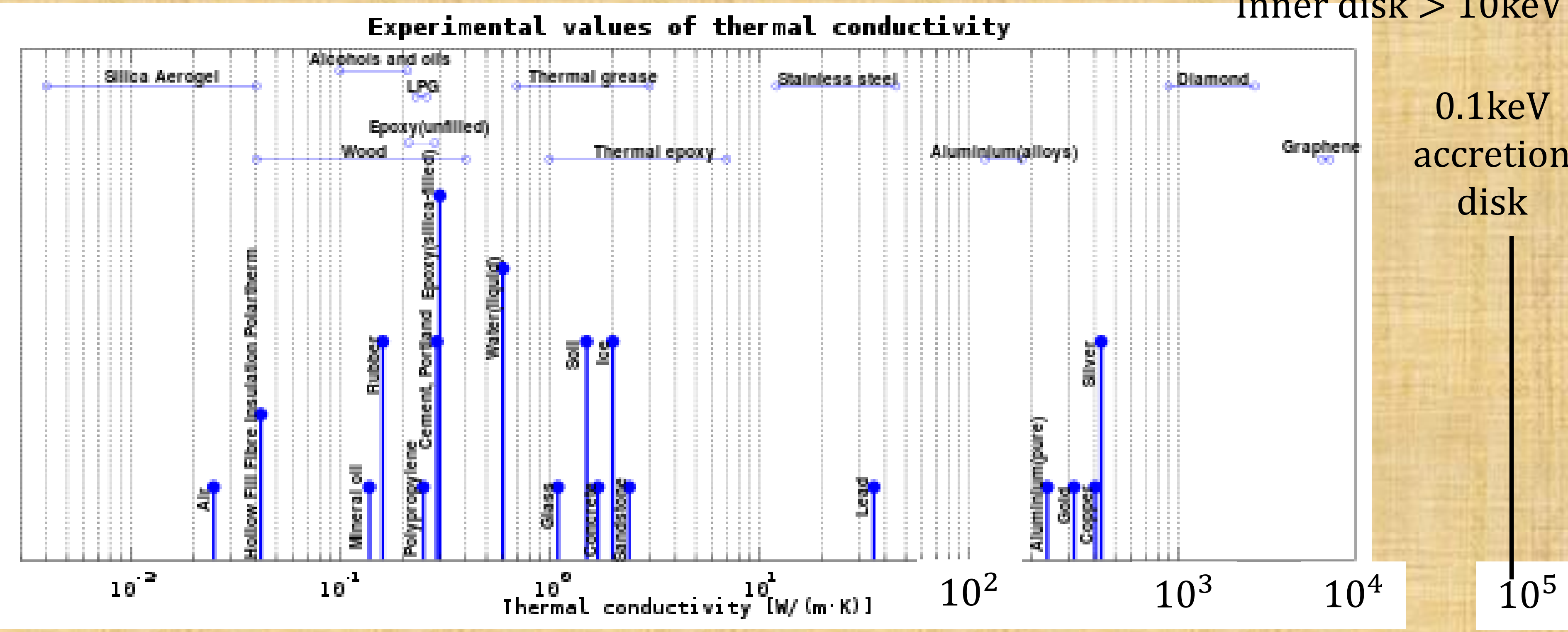
Putting it all together

The thermal conductivity $K_c \approx \frac{1}{3}(C_V n)\langle V_c\rangle \ell_c$

$$C_V = \frac{3}{2} n_e k \qquad \langle V_c\rangle = \sqrt{\frac{3\mathrm{kT}}{m_e}} \qquad \ell_c = \frac{1}{n_e \sigma_c} \qquad \sigma_c = \pi {r_c}^2 \qquad r_c = \frac{e^2}{\mathrm{kT}}$$

$$K_c \approx \frac{k}{2\pi e^4}\sqrt{\frac{3}{m_e}}(\mathrm{kT})^{\frac{5}{2}} \approx 5.38\times 10^{18}\mathrm{erg\ cm}^{-1}s^{-1}K^{-1}\left(\frac{T}{4.0\times 10^9 K}\right)^{\frac{5}{2}}$$

Inner disk > 10keV



0.1keV accretion disk

How important is it?

Let's now estimate the importance of heat flux relative to energy flux by advection from neighboring fluid elements. (Advection is from the $\vec{v} \cdot \vec{\nabla}$ term)

$$\frac{\text{Heat conduction flux}}{\text{Advection energy flux}} = \frac{Q_g}{V\varepsilon} \approx \boxed{\frac{\langle V_c \rangle}{V}} \frac{\ell_c}{R}$$

Becomes close

R is the typical length scale of system. For accreting BH, it is $\sim(10 - 100)r_g$.

Case1: Main Sequence stars:

Since MS stars are in approximately in hydrostatic equilibrium, the velocity of fluid elements V will be much smaller than $\langle V_c \rangle$ the thermal velocity.

Thus, in MS stars, heat conduction is more important.

Case2: Accreting BHs:

In such cases, V, the infall velocity, can reach the sound speed $\left(c_s = \sqrt{\frac{p}{\rho}} \approx \sqrt{\frac{\mathrm{kT}}{m_p}} \right)$.

$$\frac{Q_g}{V\varepsilon} \approx \sqrt{\frac{m_p}{m_e}} \frac{(\mathrm{kT})^2}{\pi e^4 n_e R} = 1.0 \left(\frac{n_e}{5.3{\times}10^{18}\mathrm{cm}^{-3}}\right)^{-1} \left(\frac{M}{10M_\odot}\right)^{-1} \left(\frac{R}{10r_g}\right)^{-1} \left(\frac{T}{4.{\times}10^9 K}\right)^2$$

還沒做完XD

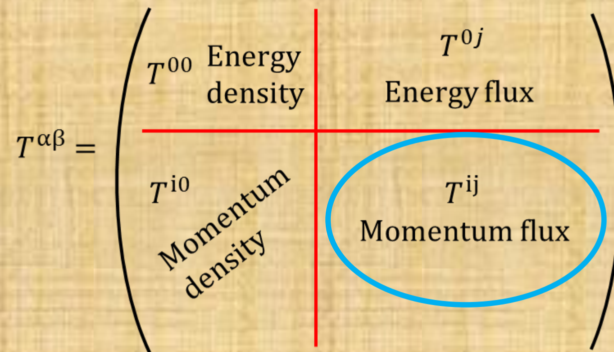
$$\frac{Q_g}{V\varepsilon} \approx \sqrt{\frac{m_p}{m_e}}\frac{(\mathrm{kT})^2}{\pi e^4 n_e R} = 1.0\left(\frac{n_e}{5.3\times 10^{18}\mathrm{cm}^{-3}}\right)^{-1}\left(\frac{M}{10M_\odot}\right)^{-1}\left(\frac{R}{10r_g}\right)^{-1}\left(\frac{T}{4.\times 10^9 K}\right)^2$$

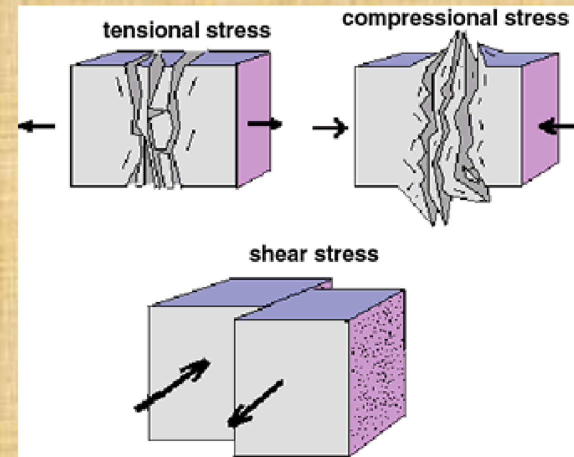
So, conduction may be more important than advection of the thermal gas for an accreting $10\,\mathrm{M}_\odot$ black hole if $n_\mathrm{e} < 10^{18}\,\mathrm{cm}^{-3}$, and for a $10^9\,\mathrm{M}_\odot$ black hole if $n_\mathrm{e} < 10^{10}\,\mathrm{cm}^{-3}$. Generally, however, in accretion situations conduction is much less important than radiative transport, so the former often can be ignored, unless the accretion flow is radiatively inefficient.

Particle Viscosity (Ch9.3.4)

Recall From last week

Since viscosity works to transport momentum, it should manifest itself in the momentum flux term of the tensor.





I'm not so familiar with this part so below mainly follows the textbook.

$$T^{\alpha\beta}{}_{\text{Viscosity}} = \underbrace{-2\eta_{v,g}\Sigma^{\alpha\beta}}_{\text{shear}} \underbrace{- \zeta_{v,g}\Theta P^{\alpha\beta}}_{\text{bulk}}$$

Projection tensor $P^{\alpha\beta} = \frac{1}{c^2}U^\alpha U^\beta + g^{\alpha\beta}$

Shear tensor $\Sigma^{\alpha\beta} \equiv \frac{1}{2}\left[P^{\alpha\gamma}\nabla_\gamma U^\beta + P^{\beta\gamma}\nabla_\gamma U^\alpha\right] - \frac{1}{3}\Theta P^{\alpha\beta}$

Compression rate $\Theta \equiv \nabla_\gamma U^\gamma$

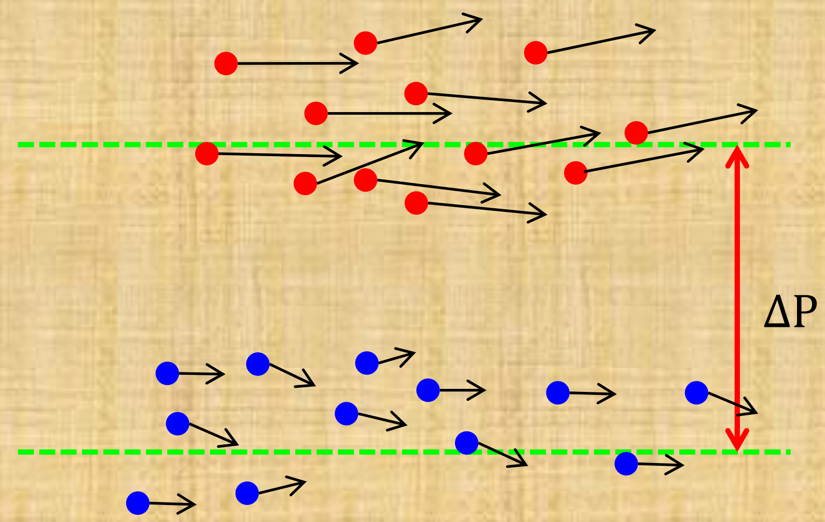
Shear viscosity coefficient

$\eta_{v,g} = \eta_{v,g}(\rho, T)$

Bulk viscosity coefficient

$\zeta_{v,g} = \zeta_{v,g}(\rho, T)$

A simple kinetic picture



Consider a picture like the one on the left.

If we consider that a pair of particles are exchanged, then there will be a net momentum transfer from top to bottom.

Therefore we can write momentum flux as (particle number flux)×(momentum difference)

In terms of differential quantities, we can write $\Delta \mathrm{P} = \ell_v m \frac{\mathrm{dV}}{\mathrm{dz}}$

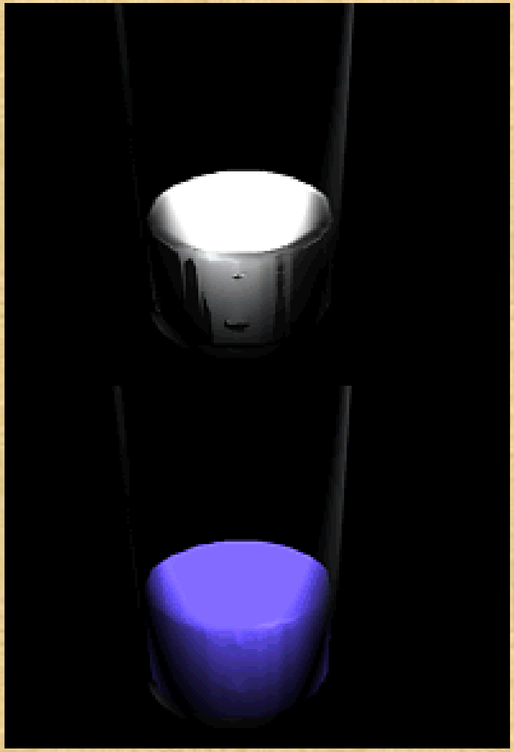
Putting it all together, we get $J_P \approx \rho \langle V_v \rangle \ell_v \frac{\mathrm{dV}}{\mathrm{dz}}$.

Comparing with ${T^{\alpha\beta}}_{\mathrm{Viscosity}} = -2\eta_{v,g}\Sigma^{\alpha\beta} - \zeta_{v,g}\Theta P^{\alpha\beta}$, we find the viscosity coefficients $\eta_{v,g} \approx \zeta_{v,g} \approx \rho \langle V_v \rangle \ell_v$

10.1098/rstl.1866.0013

The coefficients of viscosity

The coefficients of viscosity $\eta_{v,g} \approx \zeta_{v,g} \approx \rho \langle V_v \rangle \ell_v$ look very familiar to the thermal conductivity $K_c \approx \frac{1}{3}(C_V n)\langle V_c \rangle \ell_c$.



However, in case of momentum, for an electron-proton plasma, the momentum is mainly carried by the protons. Thus, both $\langle V_v \rangle$ and ℓ_v have to use values for protons.

$$\left(\ell_v = \frac{1}{n_p \sigma_c} = \frac{m_p (\mathrm{kT})^2}{\rho \pi e^4} \right)$$

$$(kT)^2 \, m_p / \rho \, \pi \, c^4$$

Typo in textbook?

Thus, for $e^- p^+$ plasma,

$$\eta_{v,g} \approx \frac{1}{\pi e^4} \sqrt{3 m_p} (\mathrm{kT})^{\frac{5}{2}} \approx 3.\times 10^9 \mathrm{erg} \cdot s \cdot \mathrm{cm}^{-3} \left(\frac{T}{4.\times 10^9 K} \right)^{\frac{5}{2}}$$

How important is it?

Comparing the contribution of viscosity to pressure, we get the following equation

$$\frac{(T^{\alpha\beta})_{\text{Visc}}}{p} \approx \frac{\eta_{v,g}\langle V_v\rangle/R}{\text{nkT}} \approx 3\,\frac{\ell_v}{R} = 3\,\frac{(\text{kT})^2}{\pi\text{e}^4\text{nR}}$$

$$= 1.0\left(\frac{n}{3.7\times10^{17}\,\text{cm}^{-3}}\right)^{-1}\left(\frac{M}{10M_\odot}\right)^{-1}\left(\frac{R}{10r_g}\right)^{-1}\left(\frac{T}{4.\times10^{9}K}\right)^{2}$$

$$\updownarrow$$

$$\frac{Q_g}{V\varepsilon} \approx \sqrt{\frac{m_p}{m_e}}\,\frac{(\text{kT})^2}{\pi e^4 n_e R} = 1.0\left(\frac{n_e}{5.3\times10^{18}\,\text{cm}^{-3}}\right)^{-1}\left(\frac{M}{10M_\odot}\right)^{-1}\left(\frac{R}{10r_g}\right)^{-1}\left(\frac{T}{4.\times10^{9}K}\right)^{2}$$

There is an interesting thing here, when we compare with the ratio of heat flux to advection energy transfer, we find that for viscosity to dominate pressure requires even lower pressure than for heat conduction to dominate advection!

Nevertheless, in low accretion rate situations, it is possible that particle viscosity may be important as well.

Turbulent Viscosity (Ch9.3.5)

呵呵呵... 看不懂啦XD

Turbulence is the random, chaotic, motion that often occurs in in fluids on scales much smaller than the overall system size, but much larger than the distance between independent fluid particles or even between fluid elements. The phenomenon usually develops in fluids undergoing shear flow, unless the microscopic viscosity is strong enough to damp out any growing chaotic motions and turn them into heat. It also can occur in fluids that have a weak, but non-zero, magnetic field. Turbulence actually is produced by fluid motions and is not really a separate physical microscopic process. However, the motions are so complex, compared to regular laminar flow that statistical methods have been developed to treat chaotic fluid motions, just as such methods were developed to handle the mechanics of multiple particles in motion in the first place.

If the size scale of the turbulent eddies ℓ_t is much smaller than the overall size of the system, then one can use the turbulent diffusion approximation to define a turbulent viscosity

$$\eta_{v,t} \approx \zeta_{v,t} \approx \rho \langle V_t \rangle \ell_t$$

where $\langle V_t \rangle$ is the RMS velocity of the chaotic motions in the eddies.

呵呵呵... 看不懂啦XD

In the early 1970s the nature of turbulent flow in black hole accretion flows was largely unknown. So early investigators [351, 352] assumed that the RMS turbulent velocity was a fraction of the sound speed

$$\langle V_t \rangle \approx \alpha c_s$$

where the free parameter $\alpha \leq 1$. This "alpha model" of turbulence was quite successful in the early days of black hole accretion studies. See Chapter 12.

In addition to the obvious issues associated with choosing a diffusion approximation, treating turbulence as a viscous process has some other assumptions associated with it. Recall (Section 9.2.1) that having a viscosity implies that viscous dissipation of the shear exists (viscous heating). This is because the viscous part of the stress-energy tensor does not have its own energy density (ε t is missing). In reality, however, turbulence does have an energy density, a pressure also, and heat flow as well, not just viscous-like properties. All of this physics is missing in this treatment, along with a model for how to convert turbulent energy into heat. Instead, the simple viscous approximation assumes that all mechanical energy lost due to viscosity immediately is converted into heat (equation (9.16)). While this works rather well in accretion models, it still should be remembered that turbulence can be much more complex than a simple ad hoc viscosity.

Radiative Opacity (Ch9.3.6)

Recall from last week

In many situations that we will study in the next few chapters, the fluid will be optically thick to radiation and both will be in thermodynamic equilibrium at the same temperature $T_r = T_g \equiv T$.

In this case the photon gas will contribute to the fluid plasma pressure, energy density, heat conduction, and viscosity and will add stress-energy terms similar to those discussed previously for fluids.

$$(T^{\alpha\beta})_{gas} = \begin{pmatrix} \underline{\rho c^2} + \underline{\varepsilon_g} & \underline{Q_g^x} & \underline{Q_g^y} & \underline{Q_g^z} \\ \underline{Q_g^x} & -2\underline{\eta_{v,g}}\Sigma^{xx} - \underline{\zeta_{v,g}}\Theta + \underline{p_g} & -2\underline{\eta_{v,g}}\Sigma^{xy} & -2\underline{\eta_{v,g}}\Sigma^{xz} \\ \underline{Q_g^y} & -2\underline{\eta_{v,g}}\Sigma^{yx} & -2\underline{\eta_{v,g}}\Sigma^{yy} - \underline{\zeta_{v,g}}\Theta + \underline{p_g} & -2\underline{\eta_{v,g}}\Sigma^{yz} \\ \underline{Q_g^z} & -2\underline{\eta_{v,g}}\Sigma^{zx} & -2\underline{\eta_{v,g}}\Sigma^{zy} & -2\underline{\eta_{v,g}}\Sigma^{zz} - \underline{\zeta_{v,g}}\Theta + \underline{p_g} \end{pmatrix}$$

$\rho = \rho_g$	Total density of fluid (photons don't contribute to this)
$p = p_g + p_r$	Total pressure
$\varepsilon = \varepsilon_g + \varepsilon_r$	Total energy density
$Q^\alpha = {Q_g}^\alpha + {Q_r}^\alpha$	Total heat conduction vector
$\eta_v = \eta_{v,g} + \eta_{v,r}$	Total coefficient of shear viscosity
$\zeta_v = \zeta_{v,g} + \zeta_{v,r}$	Total coefficient of bulk viscosity

Conduction by radiation

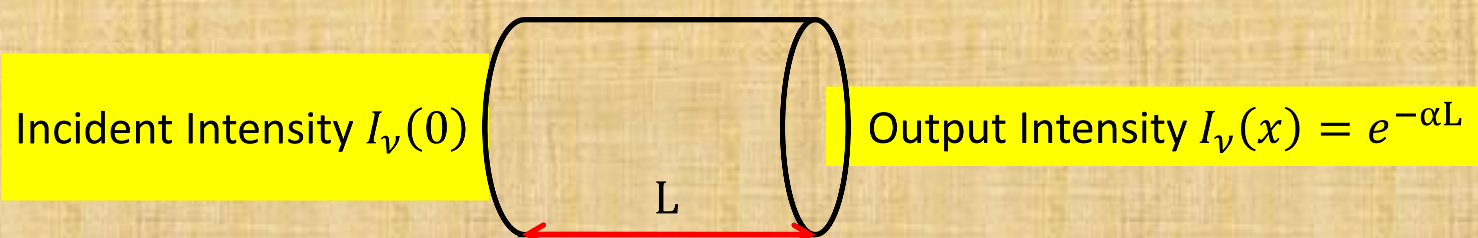
Last week, we mentioned that for radiation, we can basically copy the whole set of stress-energy tensor, therefore, for the heat conduction term, $Q^{\alpha} = {Q_g}^{\alpha} + {Q_r}^{\alpha}$.

Thus, we can determine the total conductivity $K = K_c + K_r$.

Then, by analogy of $K_c \approx \frac{1}{3}(C_V n)\langle V_c \rangle \ell_c$,

$$K_r = \frac{1}{3}(\mathrm{nC}_v)\langle V_r \rangle \ell_r = \frac{1}{3}(4\mathrm{aT}^3)c\ell_r = \frac{4\mathrm{acT}^3}{3}\frac{1}{\rho\bar{\kappa}_R}$$

$\frac{1}{\rho\bar{\kappa}_R} = \frac{1}{\alpha} = \ell_r$, α is the absorption coefficient. From illustration below, we can see that it should be inverse proportional to the mean free path of photons.



For details, please see Radiative Processes in Astrophysics by Rybicki & Lightman

Frequency dependent Opacity

Using the relations $\ell_r = \frac{1}{\alpha} = \frac{1}{\rho \bar{\kappa}_R(\nu)} = \frac{1}{n\sigma_r(\nu)}$

We can rewrite the opacity in terms of scattering/absorption coefficient

$$\bar{\kappa}_R(\nu) = \frac{n\sigma_r(\nu)}{\rho} = \sigma_r(\nu)\frac{N_A}{\mu}$$

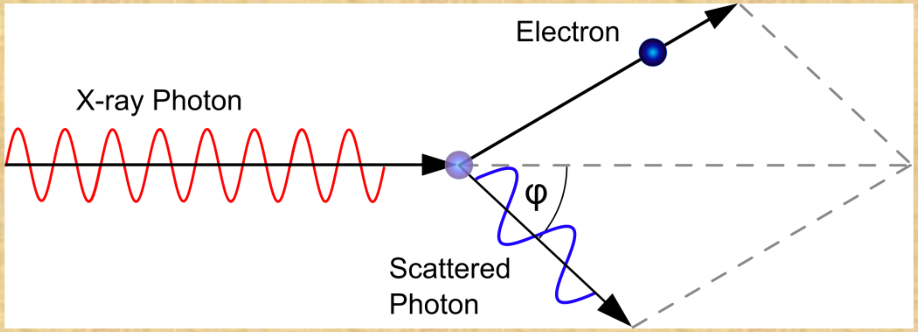
There are many ways to scatter/absorb photons. Therefore in the following we will consider

a. Electron scattering
b. Free-Free and Bound-Free Absorption

Electron scattering (Ch9.3.6.1)

General considerations

Considering scattering between photons and electrons, we recall from high school that the most general case for scattering is Compton scattering.



In such a case, the cross section we need to consider is the Klein-Nishina cross-section σ_{KN}.

Using the relation $\bar{\kappa}_R(\nu) = \frac{n\sigma_r(\nu)}{\rho} = \sigma_r(\nu)\frac{N_A}{\mu}$ and applying it to electron scattering, we find $\bar{\kappa}_R(\nu) = \sigma_{KN}(\nu)\frac{N_A(1+X)}{2}$

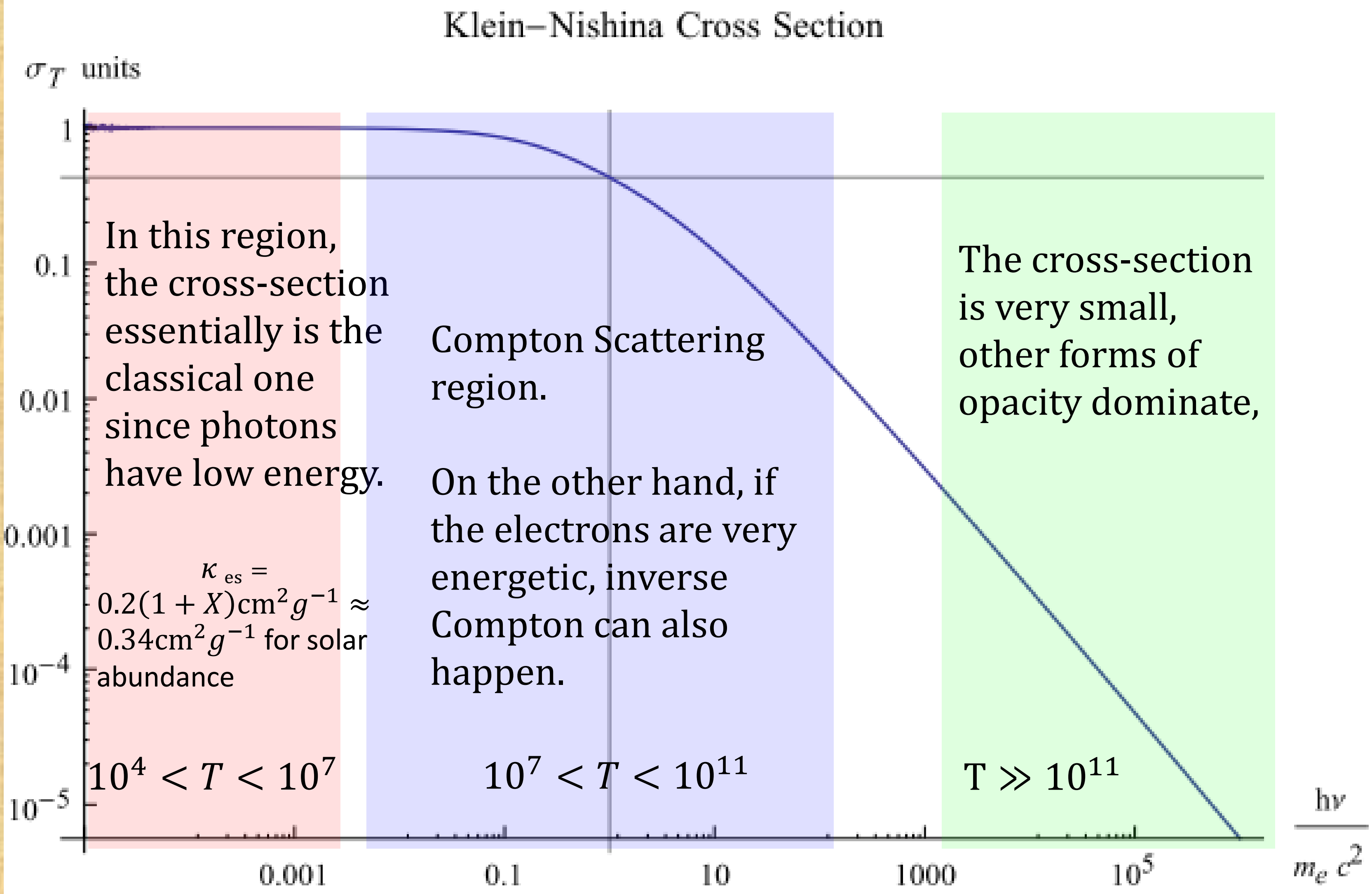
$$\sigma_{\mathrm{KN}}(\nu) = \frac{3}{4}\sigma_T\left\{\frac{2(1+\theta_\nu)}{\theta_\nu^2}\left[\frac{1+\theta_\nu}{1+2\theta_\nu} - \frac{\ell\mathrm{n}(1+2\theta_\nu)}{2\theta_\nu}\right] + \frac{\ell\mathrm{n}(1+2\theta_\nu)}{2\theta_\nu} - \frac{1+3\theta_\nu}{(1+2\theta_\nu)^2}\right\}$$

$\theta_\nu \equiv \frac{\mathrm{h}\nu}{m_e c^2} \approx \frac{T}{5.9\times10^9 K}$ is the energy of the photon in electron rest mass units

$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65246 \times 10^{-25}\mathrm{cm}^2$ is the Thomson cross-section

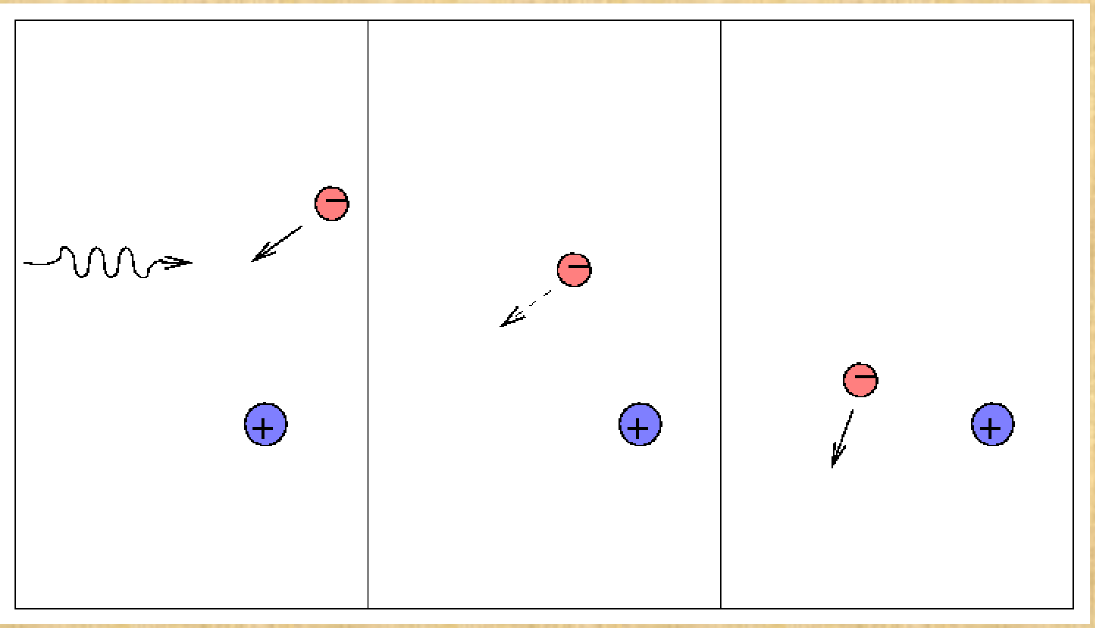
The Klein-Nishina Cross-Section

$$\sigma_{\rm KN}(\nu) = \frac{3}{4}\sigma_T \left\{ \frac{2(1+\theta_\nu)}{\theta_\nu^2}\left[\frac{1+\theta_\nu}{1+2\theta_\nu} - \frac{\ell{\rm n}(1+2\theta_\nu)}{2\theta_\nu}\right] + \frac{\ell{\rm n}(1+2\theta_\nu)}{2\theta_\nu} - \frac{1+3\theta_\nu}{(1+2\theta_\nu)^2} \right\}$$



Absorption processes (Ch9.3.6.2)

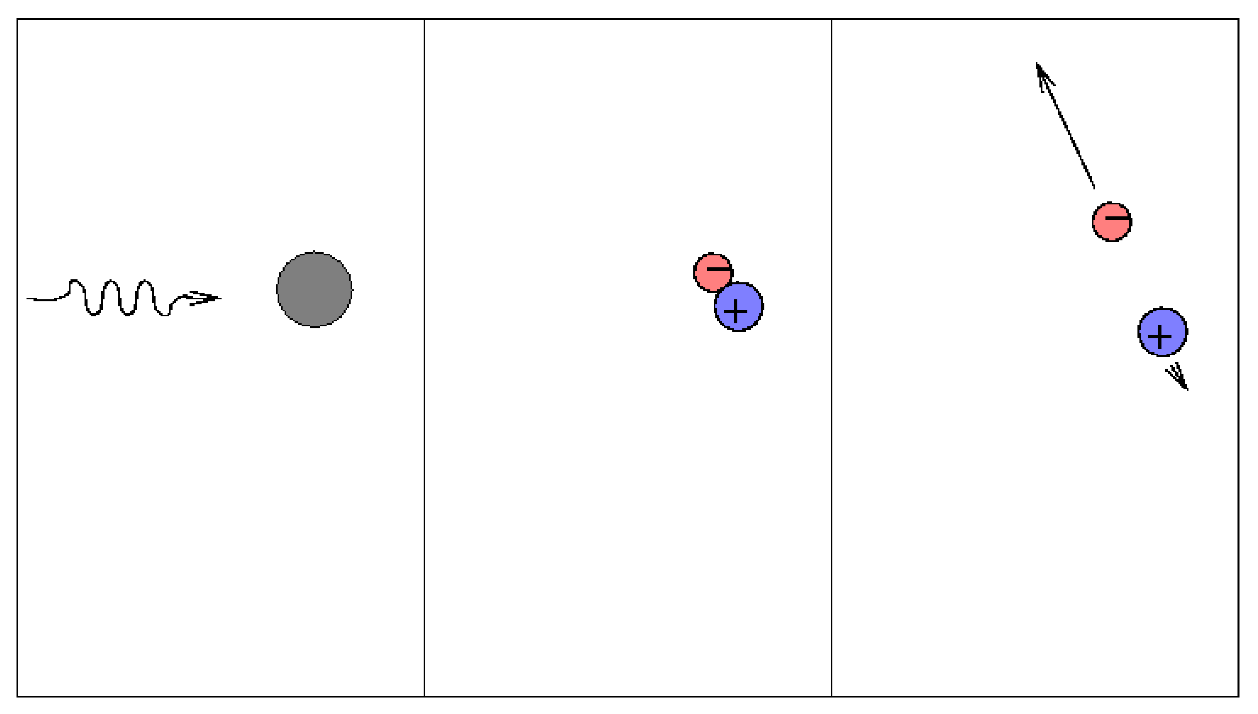
Free-Free Absorption – an Introduction



If a photon and an unbound electron collide near a positively charged ion, it is possible for the photon to be absorbed, rather than simply scattered. This process is called free–free absorption. The electron's kinetic energy increases, and, when it eventually collides with another electron or ion, that extra energy will heat the plasma.

Only much later may the inverse process (Bremsstrahlung emission), Section 9.4.1, or some other process, emit a photon again and convert that absorbed energy back into radiation. Photo-absorption and photo-emission, therefore, are treated as separate heating and cooling processes, rather than two parts of a single scattering.

Bound-Free Absorption – an Introduction



A similar effect occurs if the electron is bound to a nucleus, but the incoming photon has enough energy to eject the electron from that nucleus and ionize it. The photon again is absorbed in the event, so this process is called bound–free absorption.

The inverse process, recombination emission, also occurs separately from bound–free absorption, and need not involve the electron and ion that participated in the original ionization.

The opacities

Because free-free and bound-free are very important in stellar structure, their Rosseland means have been worked out and well known.

$$\bar{\kappa}_{R,\mathrm{ff}} = 7.36 \times 10^{22} \mathrm{cm}^2 g^{-1} (X+Y) \rho T^{-7/2} \, \bar{g}_{\mathrm{ff}} / \mu_e$$

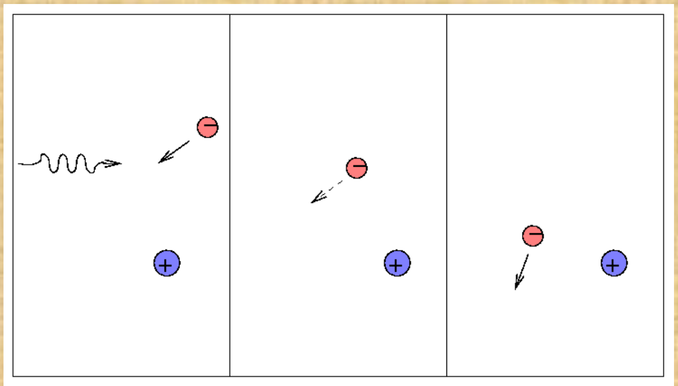
Mainly dominated by H, and He

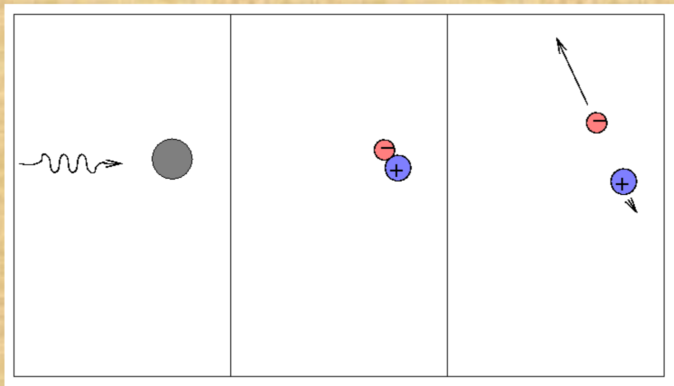
$$\bar{\kappa}_{R,\mathrm{bf}} = 8.68 \times 10^{25} \mathrm{cm}^2 g^{-1} f(T) Z \rho T^{-7/2} \, \bar{g}_{\mathrm{bf}} / \mu_e$$

Mainly dominated by heavy elements

$\bar{g}_{\mathrm{ff}}$ and $\bar{g}_{\mathrm{bf}}$ are called the Gaunt factors which are generally a factor of unity.

f(T) is the fraction of heavy elements that are not ionized $f(T) \rightarrow 0 \; as \; T \rightarrow \infty$





Total Opacity

The total opacity due to both processes is, ignoring the Gaunt factors,

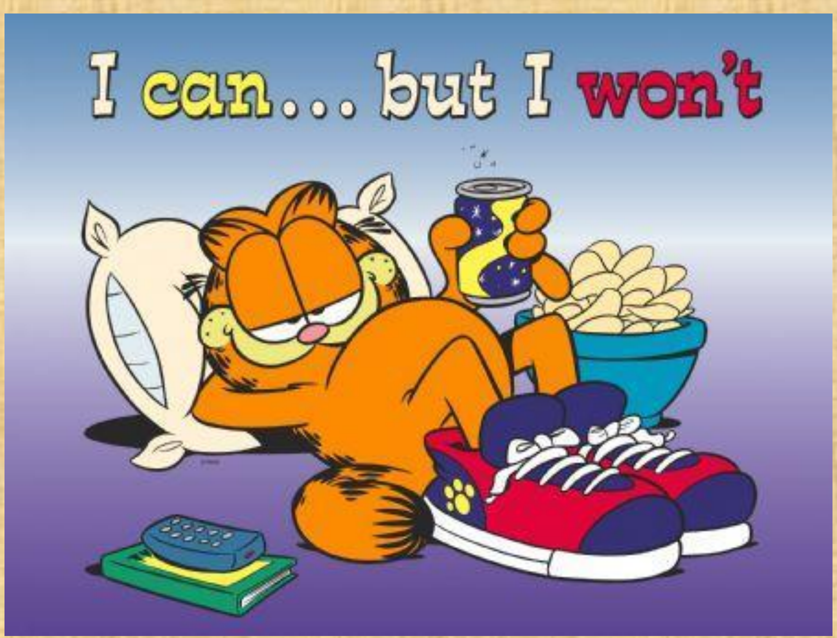
$$\bar{\kappa}_{R,\mathrm{ff}} + \bar{\kappa}_{R,\mathrm{bf}} \approx 3.68 \times 10^{22}\,\mathrm{cm}^2\,\mathrm{g}^{-1}\,[X + Y + 1180\,Z f(T)]\,(1 + X)\,\rho T^{-7/2}$$

or

$$\boxed{\bar{\kappa}_{R,\mathrm{ff}} + \bar{\kappa}_{R,\mathrm{bf}} \approx 1.55 \times 10^{24}\,\mathrm{cm}^2\,\mathrm{g}^{-1}\,\rho T^{-7/2}} \qquad (9.81)$$

for solar abundances and $f(T) \approx 1$.

Again, I am too lazy to type these equations…



Radiative Heat Transport v.s. Thermal Conduction (Ch9.3.7)

Now that we have discussed both thermal conduction and radiative heat transport, it would be interesting to see in what cases which dominate.

By taking the ratio

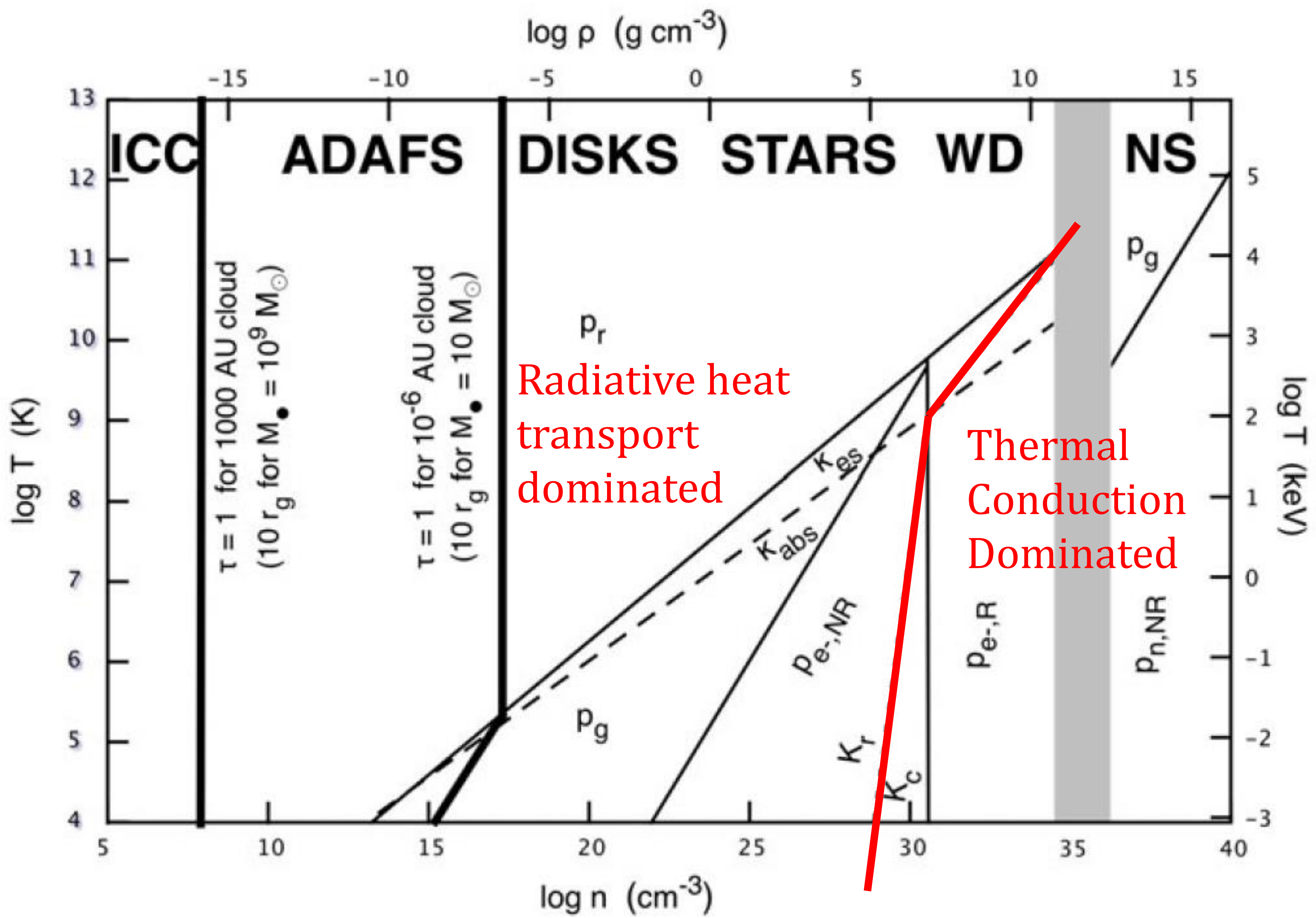
$$\frac{Q_g}{Q_r} = \frac{K_c}{K_r} = \frac{4cT^3}{\frac{\partial \varepsilon}{\partial T}\langle V_c \rangle \ell_c \bar{\kappa}_R \rho}$$

Because the opacity can be either electron scattering or absorption, we must test both cases. They give, respectively, the following criteria for *radiative* heat transport to dominate over conduction

$$n < 6.4 \times 10^{30}\,\mathrm{cm}^{-3} \left(\frac{T}{4 \times 10^9\,\mathrm{K}}\right)^{1/2} \qquad \bar{\kappa}_R = \kappa_{\mathrm{es}}$$

$$n < 4.1 \times 10^{31}\,\mathrm{cm}^{-3} \left(\frac{T}{4 \times 10^9\,\mathrm{K}}\right)^{2} \qquad \bar{\kappa}_R = \bar{\kappa}_{R,\mathrm{ff}} + \bar{\kappa}_{R,\mathrm{bf}}$$

Either way, these are enormous densities – much greater than the central density of the sun or other main sequence stars and approaching white dwarf densities. So, *for nearly all the applications of black hole accretion that we will encounter in this book, radiation will dominate conduction, and we can ignore the latter.*



log ρ (g cm-3)
-15
-10
-5
0
5
10
15
ICC
ADAFS
DISKS
STARS
WD
NS
τ = 1 for 1000 AU cloud
(10 rg for M• = 10^9 M☉)
τ = 1 for 10^-6 AU cloud
(10 rg for M• = 10 M☉)
pr
pg
κes
κabs
pe-,NR
Kr
Kc
pe-,R
pn,NR
Radiative heat transport dominated
Thermal Conduction Dominated
log T (K)
log T (keV)
log n (cm-3)